Digital Image Processing and Pattern Recognition

E1528

Fall 2021-2022

Lecture 8

Filtering in the Frequency Domain

INSTRUCTOR

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➢ Contents

Objectives

- Understand the meaning of frequency domain filtering, and how it differs from filtering in the spatial domain.
- Be familiar with the concepts of sampling, function reconstruction, and aliasing.
- Understand convolution in the frequency domain, and how it is related to filtering.
- Know how to obtain frequency domain filter functions from spatial kernels, and vice versa.
- Be able to construct filter transfer functions directly in the frequency domain.

➢ Objectives

- Understand why image padding is important.
- Know the steps required to perform filtering in the frequency domain.
- Understand when frequency domain filtering is superior to filtering in the spatial domain.
- Be familiar with other filtering techniques in the frequency domain, such as unsharp masking and homomorphic filtering.
- Understand the origin and mechanics of the fast Fourier transform, and how to use it effectively in image processing.

Introduction

- **Filter**: A device or material for suppressing or minimizing waves or oscillations of certain frequencies.
- **Frequency**: The number of times that a periodic function repeats the same sequence of values during a unit variation of the independent variable.
- After a brief historical introduction to the Fourier transform and its importance in image processing, we start from basic principles of function sampling, and proceed step-by-step to derive the one- and two-dimensional discrete Fourier transforms.

➢ Introduction

- ➢ Together with convolution, the Fourier transform is a staple of frequency-domain processing.
- ➢ During this development, we also touch upon several important aspects of sampling, such as aliasing, whose treatment requires an understanding of the frequency domain and thus are best covered in this part.
- ➢ This material is followed by a formulation of filtering in the frequency domain, paralleling the spatial filtering techniques discussed previously.

➢ Introduction

- We conclude the derivation of the equations underlying the fast Fourier transform (FFT) and discuss its computational advantages.
- These advantages make frequency-domain filtering practical and, in many instances, superior to filtering in the spatial domain.

A Brief History of the Fourier Series and Transform

- The French mathematician Jean Baptiste Joseph Fourier was born in 1768 in the town of Auxerre

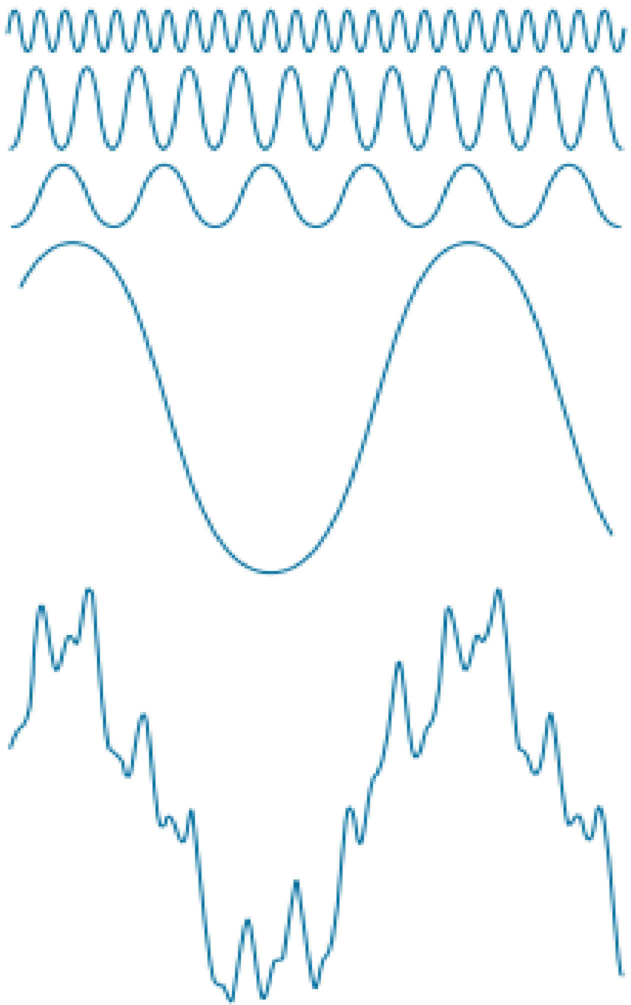
- Basically, Fourier's contribution in this field states that any periodic function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient (we now call this sum a Fourier series).

- It does not matter how complicated the function is; if it is periodic and satisfies some mild mathematical conditions, it can be represented by such a sum.

➢ A Brief History of the Fourier Series and Transform

- The function at the bottom is the sum of the four functions above it.

- Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines.



A Brief History of the Fourier Series and Transform

- The initial application of Fourier's ideas was in the field of heat diffusion, where they allowed formulation of differential equations representing heat flow in such a way that solutions could be obtained for the first time.

- During the past century, and especially in the past 60 years, entire industries and academic disciplines have succeeded as a result of Fourier's initial ideas.

- The advent of digital computers and the "discovery" of a fast Fourier transform (FFT) algorithm in the early 1960s revolutionized the field of signal processing.

A Brief History of the Fourier Series and Transform

- As you learned in last lecture, it takes on the order of MNmn operations (multiplications and additions) to filter an $M \times N$ image with a kernel of size $m \times n$ elements.
- If kernel is separable, the number of operations is reduced to $MN(m+n)$.
- In next Section, you will learn that it takes on the order of $2MN\log_2(MN)$ operations to perform the equivalent filtering process in the frequency domain, where the 2 in front arises from the fact that we have to compute a forward and an inverse FFT.

Fourier Series

- a function f (t) of a continuous variable, t, that is periodic with a period, T, can be expressed as the sum of sines and cosines multiplied by appropriate coefficients. This sum, known as a Fourier series, has the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{j\frac{2\pi n}{T}\mathrm{t}}$$

where $c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}\,dt \qquad for\; n = 0, \pm 1, \pm 2, \ldots$

- **Obtaining the Fourier Transform of a Simple Continuous Function**

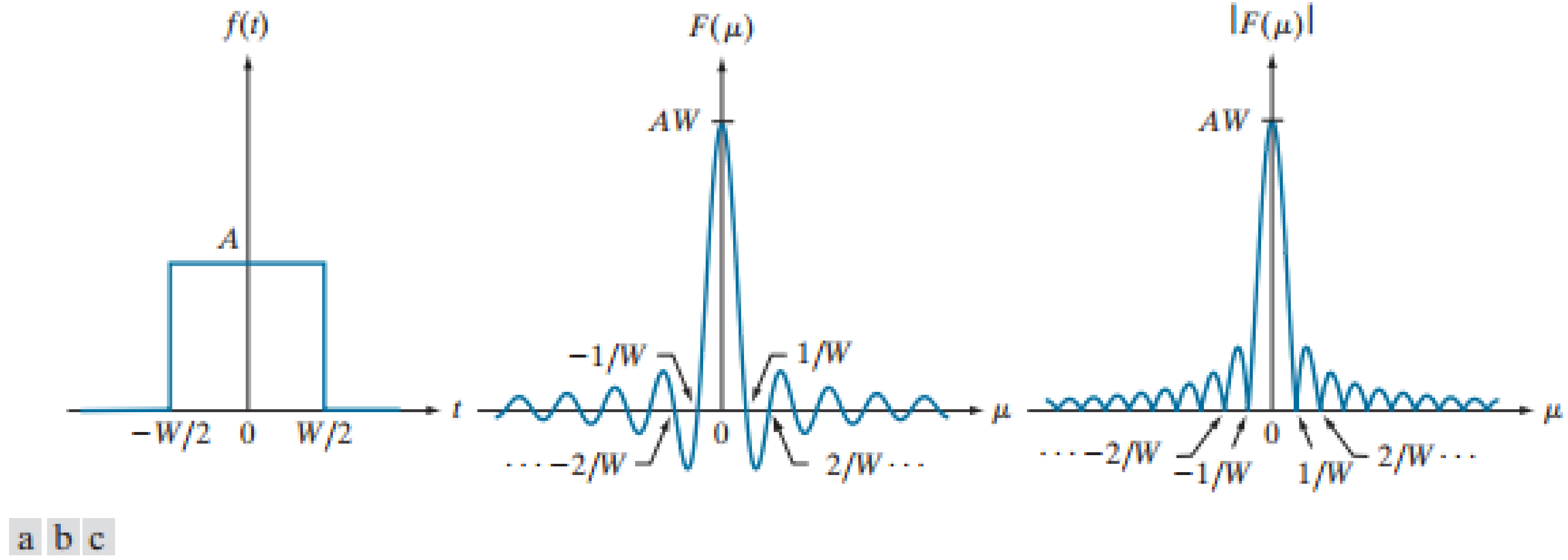
$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t} dt$$

$$= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W} - e^{j\pi\mu W}\right]$$

$$= \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right]$$

$$= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

➢ Obtaining the Fourier Transform of a Simple Continuous Function



(a) A box function, (b) its Fourier transform, and (c) its spectrum. All functions extend to infinity in both directions.

Sampling and the Fourier Transform of Sampled Functions

- In this section, we use the concepts from previous Section to formulate a basis for expressing sampling mathematically.
- Starting from basic principles, this will lead us to the Fourier transform of sampled functions. That is, the discrete Fourier transform.

Sampling

- Continuous functions must be converted into a sequence of discrete values before they can be processed in a computer.
- This requires sampling and quantization.
- Consider a continuous function, $f(t)$, that we wish to sample at uniform intervals, ΔT, of the independent variable t.
- We assume initially that the function extends from $-\infty$ to ∞ with respect to t. One way to model sampling is to multiply $f(t)$ by a sampling function equal to a train of impulses ΔT units apart.

Sampling

- That is,

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

- where $\tilde{f}(t)$denotes the sampled function.
- Each component of this summation is an impulse weighted by the value of f(t) at the location of the impulse.
- The value of each sample is given by the "strength" of the weighted impulse, which we obtain by integration.

➢ Sampling

- That is, the value, f_k, of an arbitrary sample in the sampled sequence is given by

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)dt$$

$$= f(k\Delta T)$$

- where we used the sifting property of δ. This Equation holds for any integer value k = …,-2, -1, 0, 1, 2,… next Figure(d) shows the result, which consists of equally spaced samples of the original function.

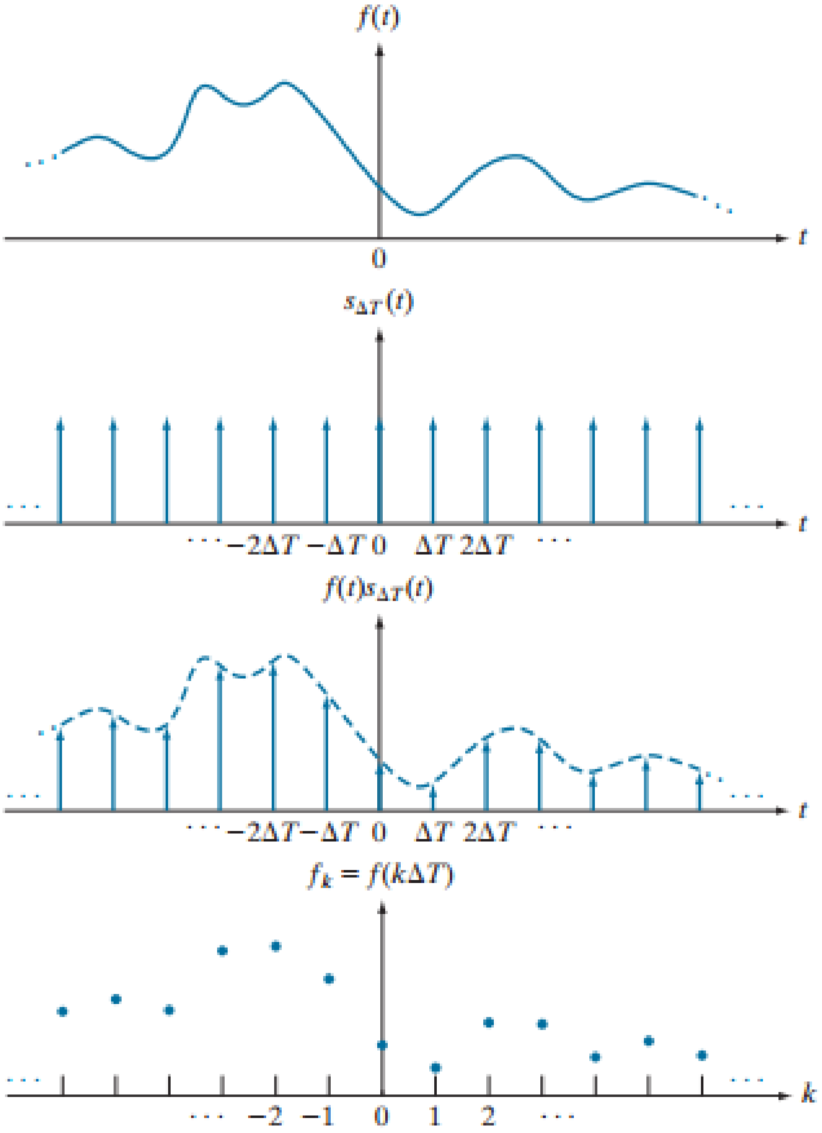
➢ Sampling

(a) A continuous function.

(b) Train of impulses used to model sampling.

(c) Sampled function formed as the product of (a) and (b).

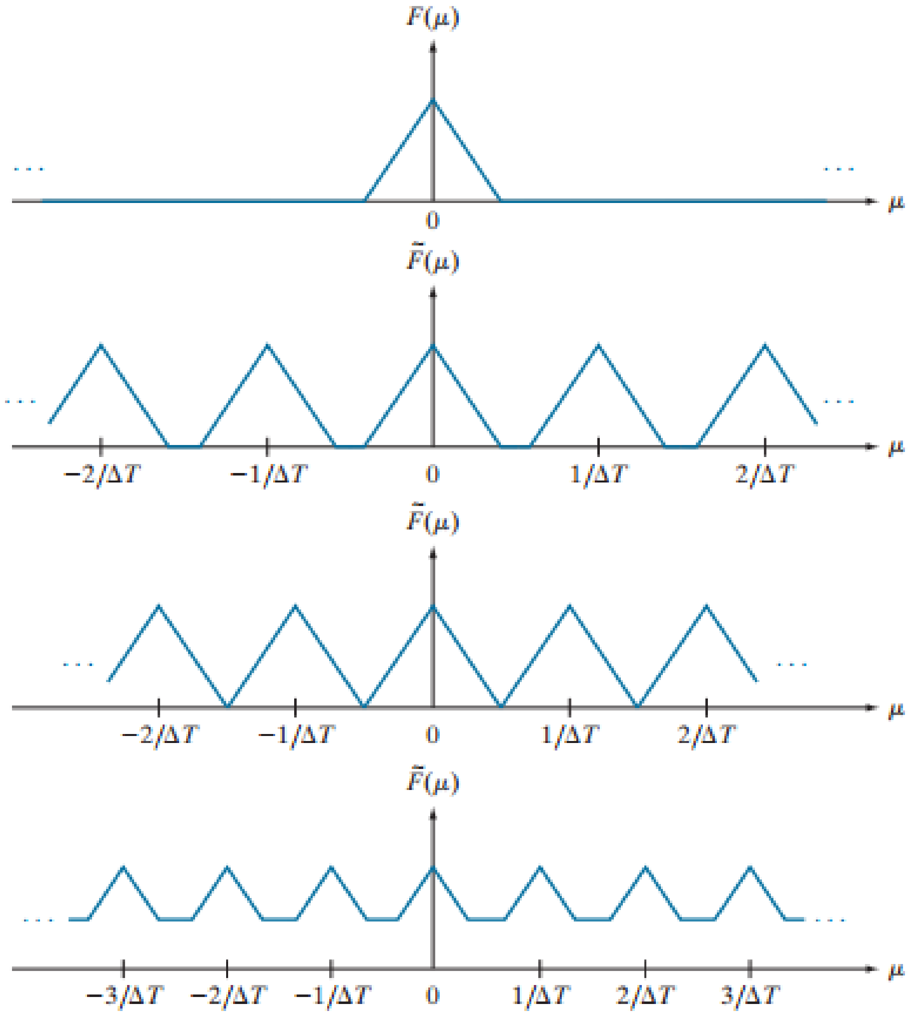
(d) Sample values obtained by integration and using the sifting property of impulses. (The dashed line in (c) is shown for reference. It is not part of the data.)



➢ The Fourier Transform of Sampled Functions

(a) Illustrative sketch of the Fourier transform of a band-limited function.

(b)–(d) Transforms of the corresponding sampled functions under the conditions of over-sampling, critically sampling, and under-sampling, respectively.



The Fourier Transform of Sampled Functions

- Extracting from $\tilde{F}(\mu)$ a single period that is equal to F(μ) is possible if the separation between copies is sufficient.
- sufficient separation is guaranteed if $1/\,2\Delta T > \mu_{max}$ or

$$\frac{1}{\Delta T} > 2\mu_{\max}$$

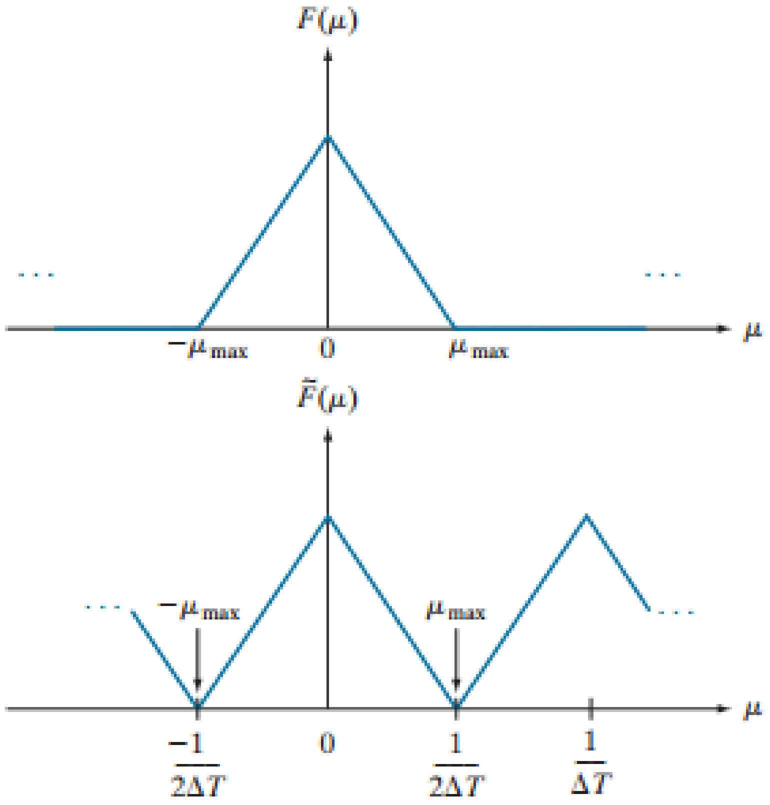
- This equation indicates that a continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceedingly twice the highest frequency content of the function.

➢ The Sampling Theorem

- ➢ A function f(t) whose Fourier transform is zero for values of frequencies outside a finite interval (band) [- μ_{max}, μ_{max}] about the origin is called a band-limited function.

(a) Illustrative sketch of the Fourier transform of a band-limited function.

(b) Transform resulting from critically sampling that band-limited function.

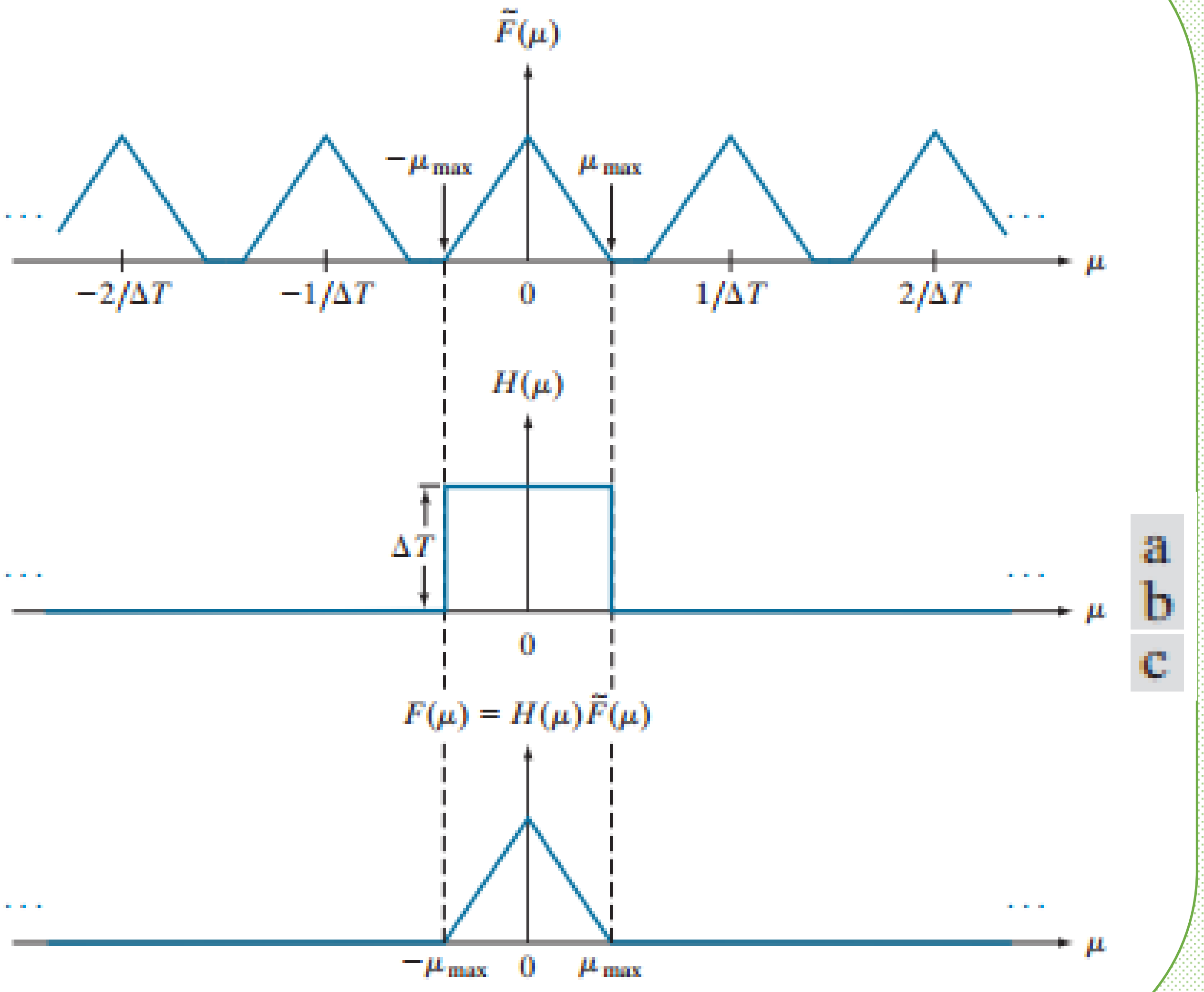


➢ The Sampling Theorem

(a) Fourier transform of a sampled, band-limited function.

(b) Ideal lowpass filter transfer function.

(c) The product of (b) and (a), used to extract one period of the infinitely periodic sequence in (a).



➢ Function Reconstruction (Recovery) from Sampled Data

- ➢ we show that reconstructing a function from a set of its samples reduces in practice to interpolating between the samples.
- ➢ Even the simple act of displaying an image requires reconstruction of the image from its samples by the display medium.
- ➢ Therefore, it is important to understand the fundamentals of sampled data reconstruction. Convolution is central to developing this understanding, demonstrating again the importance of this concept.

➢ Summary of Steps for Filtering in the Frequency Domain

1) Given an input image $f(x, y)$ of size $M \times N$, obtain the padding sizes P and Q using Equations $P=2M$ and $Q=2N$.

2) Form a padded image $f_p(x,y)$ of size $P \times Q$ using zero-, mirror-, or replicate padding.

3) Multiply fp(x,y) by $(-1)^{x+y}$ to center the Fourier transform on the $P \times Q$ frequency rectangle.

4) Compute the DFT, F(u,v) of the image from Step 3.

5) Construct a real, symmetric filter transfer function, H(u,v), of size $P \times Q$ with center at (P/2 , Q/2).

➢ Summary of Steps for Filtering in the Frequency Domain

6) Form the product G(u, v) = H (u, v) F (u, v) using elementwise multiplication; that is, G(i, k) = H (i, k) F (i, k) for i = 0, 1, 2, …, M-1 and k = 0, 1, 2, …, N-1

7) Obtain the filtered image (of size P × Q) by computing the IDFT of G(u,v):

$$g_p(x,y) = \left(\text{real}\left[\Im^{-1}\left\{G(u,v)\right\}\right]\right)(-1)^{x+y}$$

8) Obtain the final filtered result, g(x,y) of the same size as the input image, by extracting the M × N region from the top, left quadrant of $g_p(x,y)$

➢ Summary of Steps for Filtering in the Frequency Domain

- We will discuss the construction of filter transfer functions (Step 5) in the following sections.

  - In theory, the IDFT in Step 7 should be real because f (x,y) is real, and H(u,v) is real and symmetric. However, parasitic complex terms in the IDFT resulting from computational inaccuracies are not uncommon.

  - Taking the real part of the result takes care of that. Multiplication by $(-1)^{x+y}$ cancels out the multiplication by this factor in Step 3.

➢ Image Smoothing Using Lowpass Frequency Domain Filters

Types of Lowpass Filters

- *Ideal*
- *Butterworth*
- *Gaussian*

Types of Lowpass Filters

- These three categories cover the range from very sharp (ideal) to very smooth (Gaussian) filtering.
- The shape of a Butterworth filter is controlled by a parameter called the filter order.
- For large values of this parameter, the Butterworth filter approaches the ideal filter.
- For lower values, the Butterworth filter is more like a Gaussian filter.

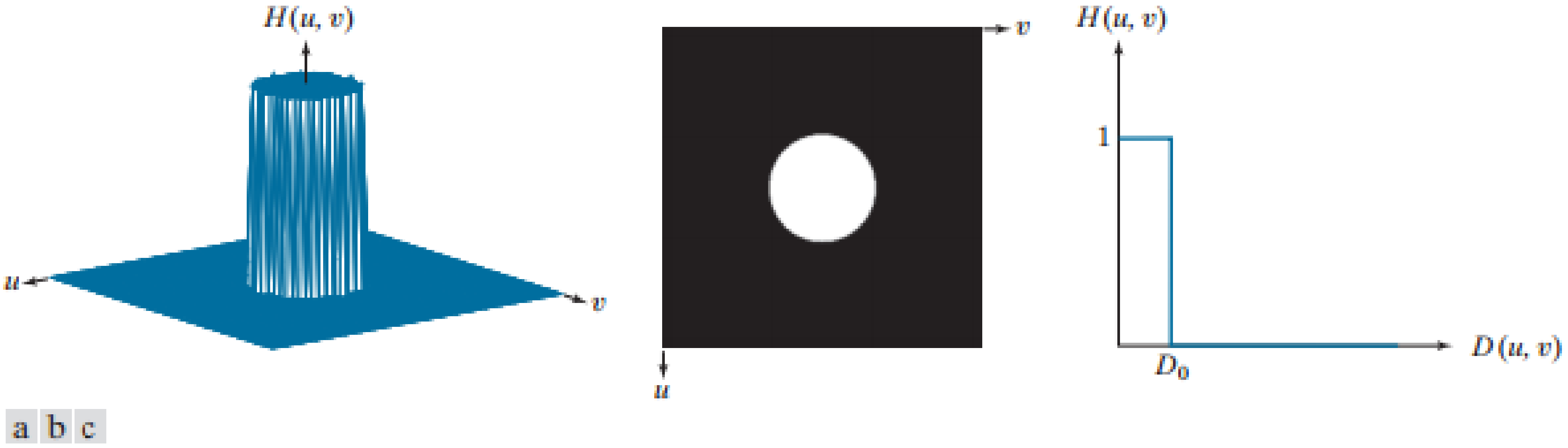
Ideal Lowpass Filters

- A 2-D lowpass filter that passes without attenuation all frequencies within a circle of radius from the origin, and "cuts off" all frequencies outside this, circle is called an ideal lowpass filter (ILPF); it is specified by the transfer function

$$H(u,v)=\begin{cases} 1 & if\ D(u,v) \leq D_0 \\ 0 & if\ D(u,v) \geq D_0 \end{cases}$$

- where D_0 is a positive constant, and D(u,v) is the distance between a point (u,v) in the frequency domain and the center of the P × Q frequency rectangle;

➢ Ideal Lowpass Filters



(a) Perspective plot of an ideal lowpass-filter transfer function.

(b) Function displayed as an image.

(c) Radial cross section.

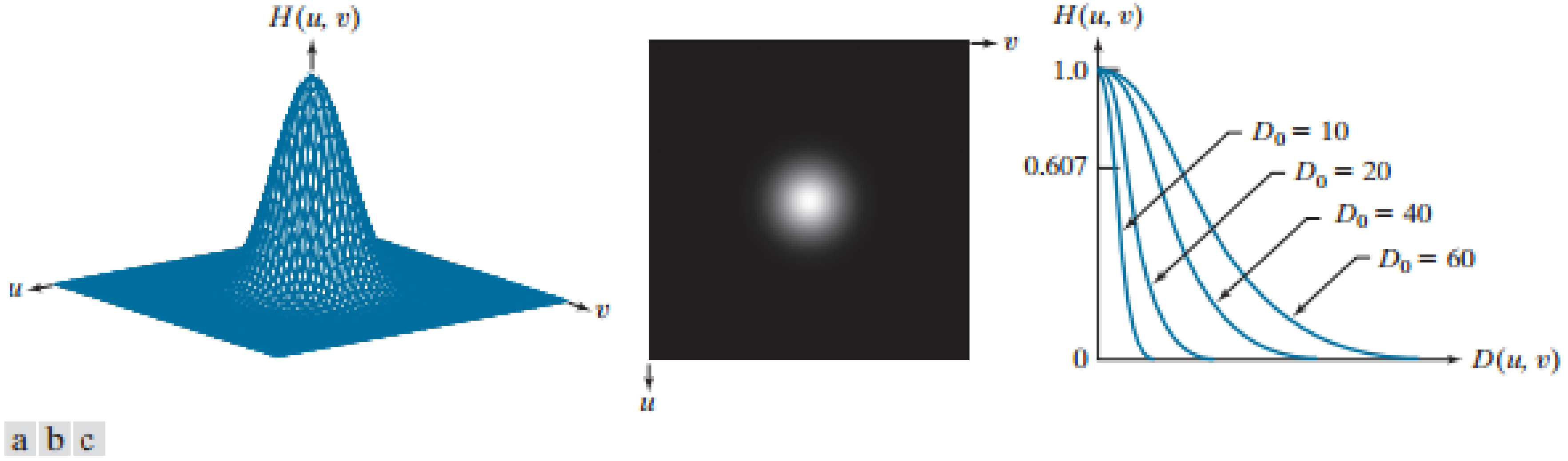
➢ Ideal Lowpass Filters

(a) Original image of size 688 × 688 pixels. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460. The power removed by these filters was 13.1, 7.2, 4.9, 2.4, and 0.6% of the total, respectively. We used mirror padding to avoid the black borders characteristic of zero padding.

➢ Gaussian Lowpass Filters

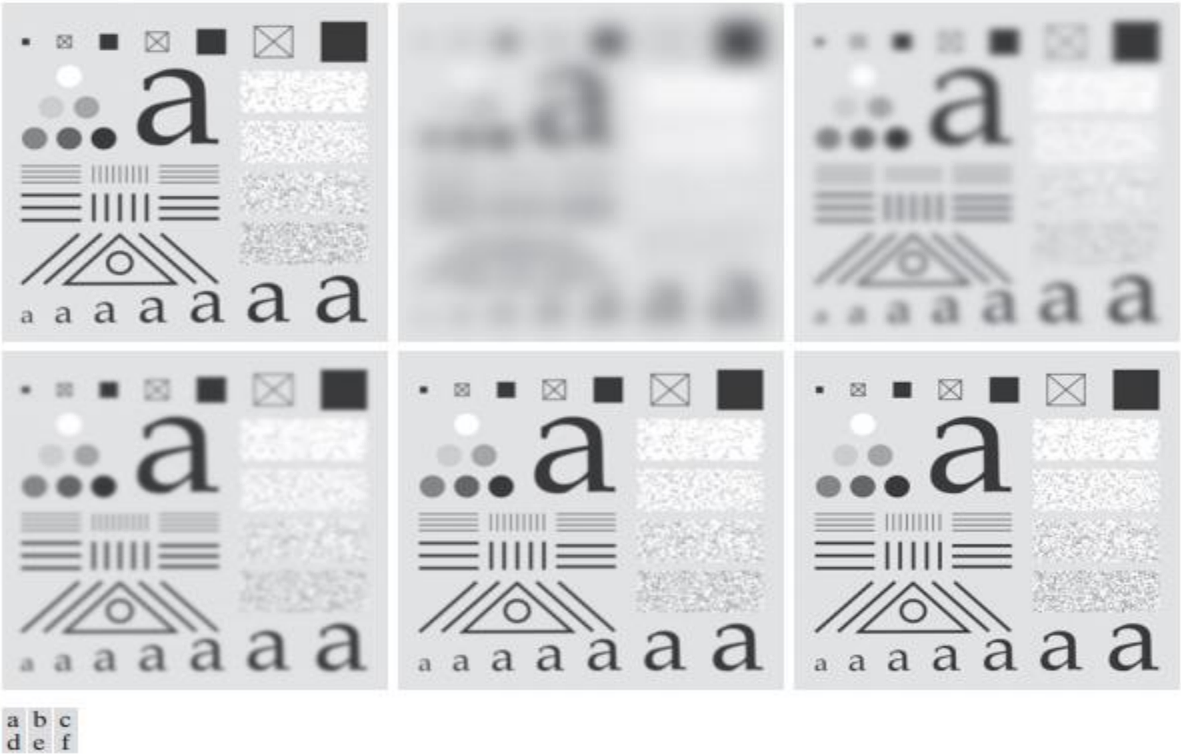
- ➢ Gaussian lowpass filter (GLPF) transfer functions have the form

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$



(a) Perspective plot of a GLPF transfer function. (b) Function displayed as an image. (c) Radial cross sections for various values of D_0.

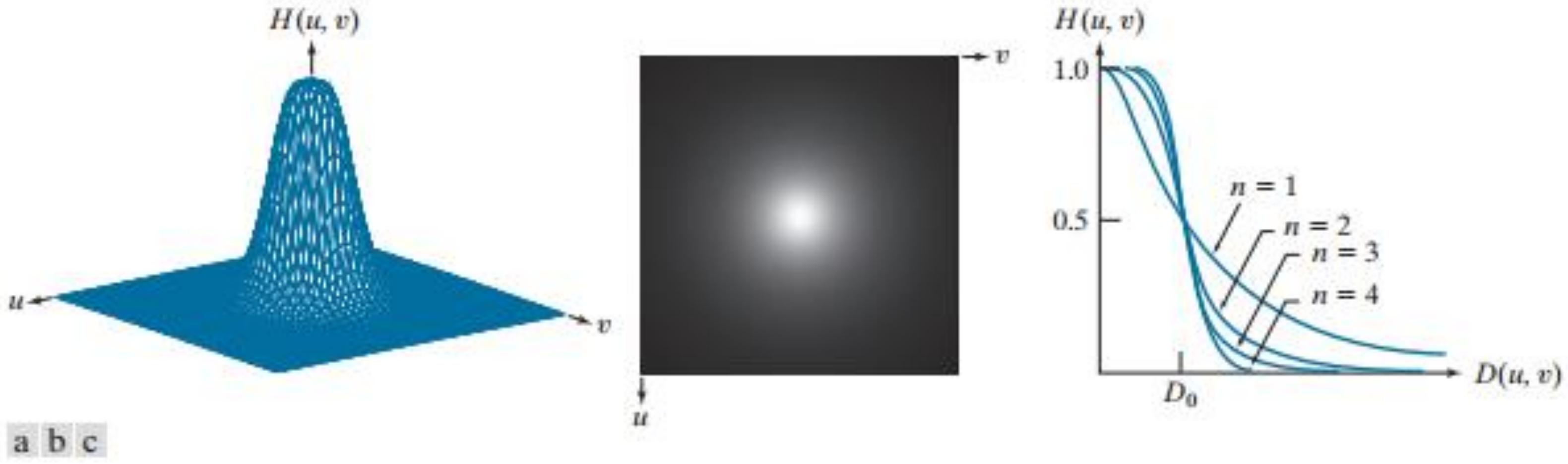
➢ Gaussian Lowpass Filters



(a) Original image of size 688 × 688 pixels. (b)–(f) Results of filtering using GLPFs with different cutoff frequencies. We used mirror padding to avoid the black borders characteristic of zero padding.

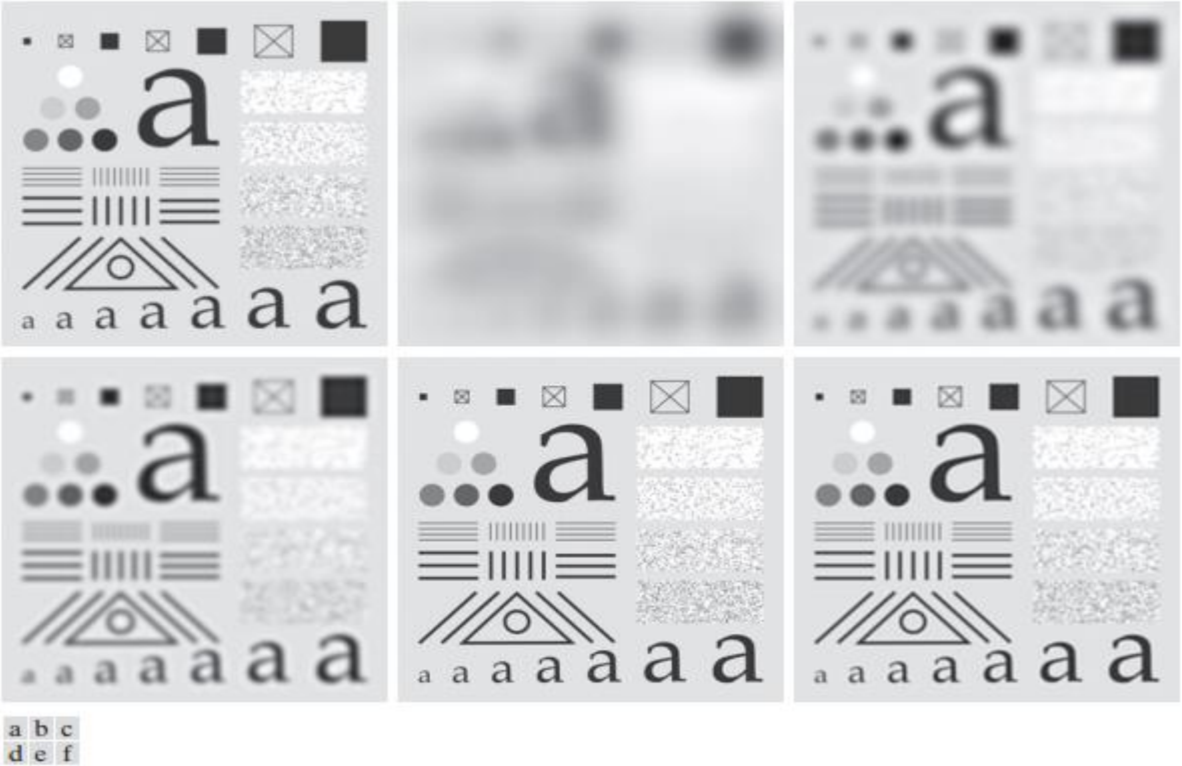
Butterworth Lowpass Filters

- The transfer function of a Butterworth lowpass filter (BLPF) of order n, with cutoff frequency at a distance D_0 from the center of the frequency rectangle, is defined as $$H(u,v) = \frac{1}{1+[D(u,v)/D_0]^{2n}}$$



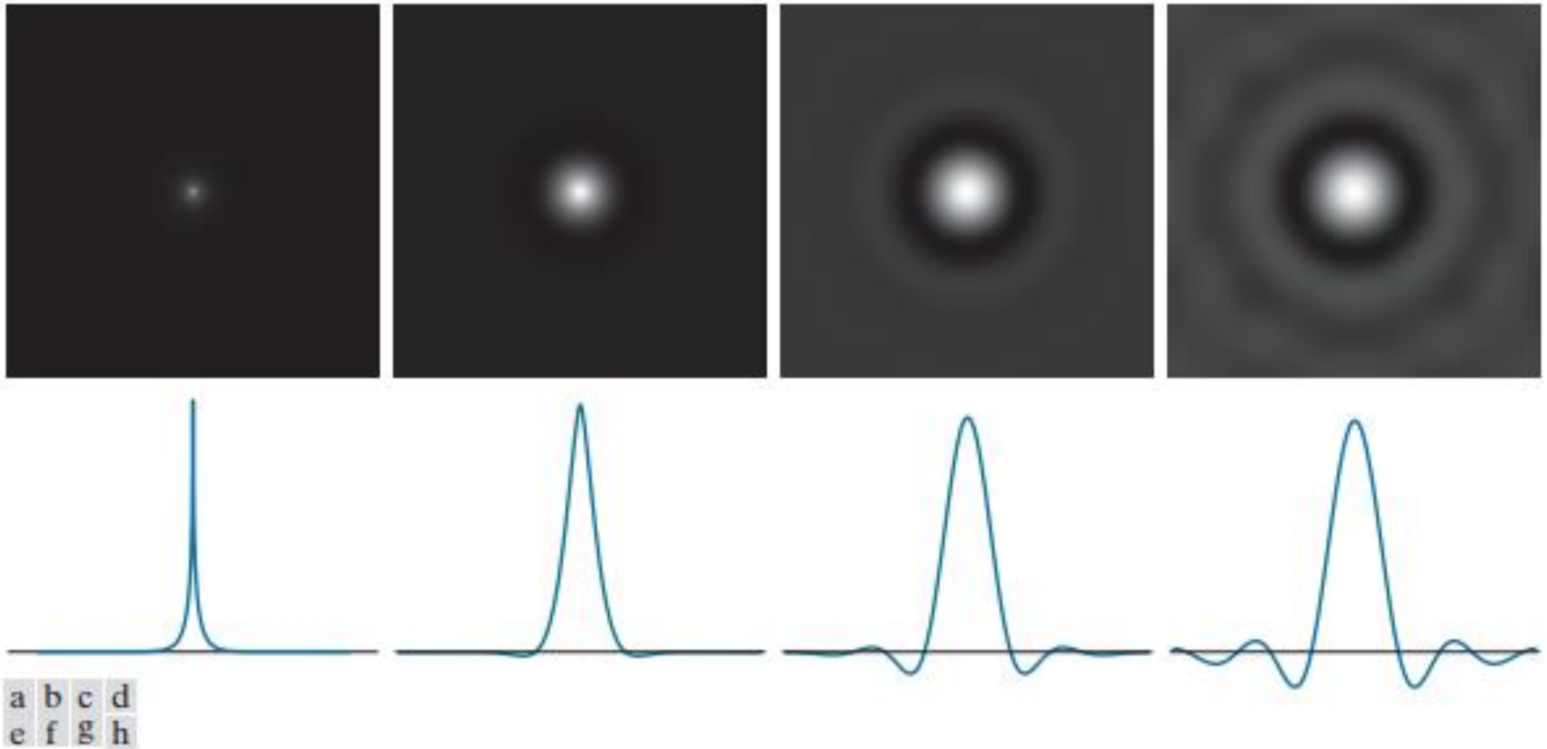
(a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Function displayed as an image. (c) Radial cross sections of BLPFs of orders 1 through 4.

➢ Butterworth Lowpass Filters



(a) Original image of size 688 × 688 pixels. (b)–(f) Results of filtering using BLPFs with cutoff frequencies at the different radii and n = 2.5. We used mirror padding to avoid the black borders characteristic of zero padding.

➢ Butterworth Lowpass Filters



(a)–(d) Spatial representations (i.e., spatial kernels) corresponding to BLPF transfer functions of size 1000 × 1000 pixels, cut-off frequency of 5, and order 1, 2, 5, and 20, respectively.

(e)–(h) Corresponding intensity profiles through the center of the filter functions.

➢ Summary

- Lowpass filter transfer functions. D_0 is the cutoff frequency, and n is the order of the Butterworth filter.

Ideal	Gaussian	Butterworth
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$	$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$


Thank you